

Discrete Mathematics(II)

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Review

- Language
- Truth table
- Connectives

Outline

- Formation tree
- Parsing algorithm

Ambiguity

Example

Consider the following sentences:

- 1 The lady hit the man with an umbrella.
- 2 He gave her cat food.
- 3 They are looking for teachers of French, German and Japanese.

Ambiguity

Example

Consider the following proposition

$$A_1 \vee A_2 \wedge A_3.$$

We have two possible different propositions

- 1 $(A_1 \vee A_2) \wedge A_3$
- 2 $A_1 \vee (A_2 \wedge A_3)$

Of course, they have different abbreviated truth tables.

Count of Parentheses

Theorem

Every well-formed proposition has the same number of left as right parentheses.

Proof.

- 1 Consider the symbols without parentheses first.
- 2 And then prove it by induction with more complicated propositions according to the Definition.



Theorem

Any proper initial segment of a well-defined proposition contains an excess of left parenthesis. Thus no proper initial segment of a well defined proposition can itself be a well defined propositions.

Proof.

Prove it by induction from simple to complicated propositions. □

Formation Tree

Example

The formation tree of $(A \vee B), ((A \wedge B) \rightarrow C)$.

How?

Form a tree bottom-up while constructing the proposition according to the Definition.

Formation Tree

Definition (Top-down)

A *formation tree* is a finite tree T of binary sequences whose nodes are all labeled with propositions. The labeling satisfies the following conditions:

- 1 The leaves are labeled with propositional letters.
- 2 if a node σ is labeled with a proposition of the form $(\alpha \vee \beta)$, $(\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$ or $(\alpha \leftrightarrow \beta)$, its immediate successors, $\sigma^{\wedge}0$ and $\sigma^{\wedge}1$, are labeled with α and β (in that order).
- 3 if a node σ is labeled with a proposition of the form $(\neg\alpha)$, its unique immediate successor, $\sigma^{\wedge}0$, is labeled with α .

Definition

- 1 The *depth* of a proposition is the depth of associated formation tree.
- 2 The *support* of a proposition is the set of propositional letters that occur as labels of the leaves of the associated formation tree.

Formation Tree

Theorem

Each well-defined proposition has a unique formation tree associated with it.

Proof.

- 1 Existence of the formation tree by induction on depth.
- 2 Uniqueness of the formation tree by induction on depth.



Parsing algorithm

- 1 Is an expression a well defined proposition?
- 2 How to check?
- 3 Build the formation tree.

Parsing algorithm

- step 1 If all leaf nodes are labeled with proposition letters, stop it. Otherwise select a leaf node having expressions other than letter and examine it.
- step 2 The first symbol must be (. if the second symbol is \neg , jump to step 3. Otherwise go to step 4.
- step 3 The first two symbols are now known to be $(\neg$. The remainder of the expression, β) must consist of a an expression β and $)$. Then we extend the tree by adding β as its immediate successor. Goto step 1.
- step 4 Next slide.

Parsing algorithm

Step 4:

- 1 Scan the expression from the left until first reaching $(\alpha$, where α is a nonempty expression having a balance between $($ and $)$.
- 2 The α is the first of the two constituents.
- 3 The next symbol must be $\wedge, \vee, \rightarrow$, or \leftrightarrow .
- 4 The remainder of the expression, β) must consist of an expression β and $)$.
- 5 Extend the tree by adding α and β as left and right immediate successor respectively.

Parentheses

Method 1: define the following conventions:

- 1 The outermost parenthesis need not be explicitly mentioned.
- 2 The negation symbol applies to a proposition as little as possible.
- 3 The conjunction and disjunction symbols also apply as little as possible.
- 4 Where a connective symbol is used repeatedly, grouping is to the right.

Parentheses

Method 2: Polish notation.

- 1 $\mathcal{D}_{\neg}(\alpha) = \neg\alpha.$
- 2 $\mathcal{D}_{\vee}(\alpha, \beta) = \vee\alpha\beta.$
- 3 $\mathcal{D}_{\wedge}(\alpha, \beta) = \wedge\alpha\beta.$
- 4 $\mathcal{D}_{\rightarrow}(\alpha, \beta) = \rightarrow\alpha\beta.$
- 5 $\mathcal{D}_{\leftrightarrow}(\alpha, \beta) = \leftrightarrow\alpha\beta.$

Example

Determine the well defined proposition of

$$\rightarrow \wedge AD \vee \neg B \leftrightarrow CB.$$

Inductive definition

Definition

A set S is *closed* under a single operation $f(s_1, \dots, s_n)$ if and only if every $s_1, \dots, s_n \in S, f(s_1, \dots, s_n) \in S$

Definition

The *closure* of a set S under (all) the operations in a set T is the smallest C such that

- 1 $S \subseteq C$ and
- 2 if $f \in T$ is n -ary and $s_1, \dots, s_n \in C$, then $f(s_1, \dots, s_n) \in C$.

Inductive definition

Theorem

Suppose that S is closed under the operations of T , there is a smallest closure C such that $S \subseteq C$.

Proof.

Given a set D , $S \subseteq D$ and D is closed under the operations of T . Consider the set

$$C = \cap \{D \mid S \subseteq D \wedge D \text{ is closed under the operations of } T\}.$$

It is obvious that $S \subseteq C$. And C is the smallest set. \square

Inductive definition

Theorem

If S is a set of well defined propositions and is closed under all five connectives, then S is the set of all well defined propositions

Next Class

- Assignment
- Truth value
- Deduction theorem