Discrete Mathematics

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Review

- Introduction
- Tree
- König lemma

Outline

- Propositions
- Truth table
- Adequacy

Sentences

Example

Consider the following statements:

Sentences

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Consider the following statements:

1. I am a student.

Sentences

Example

Consider the following statements:

1. I am a student.
2. I am not a student.

Sentences

Example

Consider the following statements:

1. I am a student.
2. I am not a student.
3. I am a student and I study computer science.

Sentences

Example

Consider the following statements:

1. I am a student.
2. I am not a student.
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4. I am a boy or I am a girl.

Sentences

Example

Consider the following statements:

1. I am a student.
2. I am not a student.
3. I am a student and I study computer science.
4. I am a boy or I am a girl.
5. If I am a student, I have a class in a week.

Sentences

Example

Consider the following statements:

1. I am a student.
2. I am not a student.
3. I am a student and I study computer science.
4. I am a boy or I am a girl.
5. If I am a student, I have a class in a week.
6. I am student if and only if I am a member of some university.

Sentences

We don't care about the following:

- Are you a student?

Sentences

We don't care about the following:

- Are you a student?
- Sit down please.

Sentences

We don't care about the following:

- Are you a student?
- Sit down please.
- What are you doing?

Connectives

A summary of connectives:

Symbol	Verbose name	Remark
$\vee$	disjunction	or
$\wedge$	conjunction	and
$\neg$	negation	not
$\rightarrow$	conditional	if ..., then ...
$\leftrightarrow$	biconditional	if and only if

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  2. Parentheses:), (
  3. Propositional Letters: $A, A_1, A_2, \cdots, B, B_1, B_2, \cdots$.
- A propositional letter is the most elementary object.

Propositions

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Propositions

Definition (Proposition)

1. Propositional letters are propositions.
2. if α and β are propositions, then $(\alpha \vee \beta), (\alpha \wedge \beta), (\neg \alpha), (\alpha \rightarrow \beta)$ *and* $(\alpha \leftrightarrow \beta)$ are propositions.
3. A string of symbols is a proposition if and only if it can be obtained by starting with propositional letters (1) and repeatedly applying (2).

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1. $(A \vee B), ((A \wedge B) \rightarrow C)$.

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Example

Check the following strings:

1. $(A \vee B), ((A \wedge B) \to C)$.
2. $A \vee \neg, (A \wedge B$

Truth Tables

α	β	$\alpha \vee \beta$
T	T	T
T	F	T
F	T	T
F	F	F

α	β	$\alpha \wedge \beta$
T	T	T
T	F	F
F	T	F
F	F	F

α	β	$\alpha \leftrightarrow \beta$
T	T	T
T	F	F
F	T	F
F	F	T

α	$\neg\alpha$
T	F
F	T

Truth Tables

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
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Truth Tables

Why do we let $\alpha \to \beta$ true when α is false?

Example

Consider the proposition, if $n > 2$, then $n^2 > 4$.

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Solution.

We first all know that the statement is correct. Let $n = 3, 1, -3$. Consider the truth of the statement:

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1. $n = 3$, true and true.

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We first all know that the statement is correct. Let $n = 3, 1, -3$. Consider the truth of the statement:

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Solution.

We first all know that the statement is correct. Let $n = 3, 1, -3$. Consider the truth of the statement:

1. $n = 3$, true and true.
2. $n = 1$, false and false.
3. $n = -3$, false and true.

□

Truth Tables

Example

Figure out what would happen if man can fly like a bird!

Connectives

Definition (Truth functional)

An n-ary connective is *truth functional* if the truth value for $\sigma(A_1, \ldots, A_n)$ is uniquely determined by the truth value of $A_1, \ldots, A_n$.

Connectives

Definition (Boolean function)

A k-place *Boolean function* is a function from $\{F, T\}^k$ to $\{T, F\}$. We let F and T themselves to be 0-place Boolean functions.

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Example

x_1	x_2	$x_1 \to x_2$	$f_\to(x_1, x_2)$
T	T	T	$f_\to(T, T) = T$
T	F	F	$f_\to(T, F) = F$
F	T	T	$f_\to(F, T) = T$
F	F	T	$f_\to(F, F) = T$

Connectives

There are many connectives.

1. 0-ary connectives: T and F.

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2. Unary connectives: $\neg$, I , T and F.

Where $I_i(x_1, x_2, \ldots, x_n) = x_i$, which is a projection function of i-th parameter.

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how many n-place Boolean functions are there?

Adequacy

For each n distinct letters, there are totally 2^{2^n} n-place booleann functions.

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Definition (Adequate connectives)

A set S of truth functional connectives is *adequate* if, given any truth function connective σ, we can find a proposition built up from the connectives is S with the same abbreviated truth table as σ.

Adequacy

Theorem (Adequacy)

$\{\neg, \vee, \wedge\}$ *is adequate(complete).*

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Proof.

Construct the truth table of any connective $\sigma(A_1, \ldots, A_k)$. □

Adequacy

Corollary

$\{\neg, \vee\}$ *is adequate.*

Normal Form

Definition (DNF)

α is called *disjunctive normal form* (abbreviated DNF). If α is a disjunction

$$\alpha = \gamma_1 \vee \cdots \vee \gamma_k,$$

where each γ_i is a conjunction

$$\gamma_i = \beta_{i1} \wedge \cdots \wedge \beta_{in_i}$$

and each β_{ij} is a proposition letter or the negation of a proposition letter.

Normal Form

Example

$\alpha = (A_1 \wedge A_2 \wedge A_3) \vee (\neg B_1 \wedge B_2) \vee (\neg C_1 \wedge \neg C_2 \wedge \neg C_3)$ is a DNF.

Normal Form

Definition (CNF)

α is called *conjunctive normal form* (abbreviated CNF). If α is a conjunction

$$\alpha = \gamma_1 \wedge \cdots \wedge \gamma_k,$$

where each γ_i is a disjunction

$$\gamma_i = \beta_{i1} \vee \cdots \vee \beta_{in_i}$$

and each β_{ij} is a proposition letter or the negation of a proposition letter.

Normal Form

Example

$\alpha = (A_1 \vee A_2 \vee A_3) \wedge (\neg B_1 \vee B_2) \wedge (\neg C_1 \vee \neg C_2 \vee \neg C_3)$ is a CNF.

Normal Form

Theorem

Any proposition can be reformed as a DNF and a CNF.

How?

Normal Form

Theorem

Any proposition can be reformed as a DNF and a CNF.

How?

Proof.

According to adequacy theorem. □

Next Class

- Formation tree
- Proposition parsing