Discrete Mathematics

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Outline

- Review of partial order set
- Review of abstract algebra
- Lattice and Sublattice

Introduction

- Intensively explored area
  1. By 1960s, 1,500 papers and books
  2. By 1970s, 2,700 papers and books
  3. By 1980s, 3,200 papers and books
  4. By 1990s, 3,600 papers and books
- History
  1. By 1850, George Boole's attempt to formalize proposition logic.
  2. At the end of 19th century, Charles S. Pierce and Ernst Schröder
  3. Independently, Richar Dedekind.
  4. Until mid-1930's, Garrett Birkhoff developed general theory on lattice.

Partial order set(Poset)

Definition

Given a set A and a relation R on it, $< A, R >$ is called a partially ordered set(**poset** in brief) if R is *reflexive*, *antisymmetric* and *transitive*.

Poset

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4. a is least if for every $b \in A$, we have $a \leq b$.

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Theorem

A poset has at most one LUB or GLB.

Lattice

Definition

A *lattice* (structure) is a poset $< A, \leq >$ in which any two elements a, b have a $LUB(a, b)$ and a $GLB(a, b)$.

From now on, we define $a \cup b = LUB(a, b)$ and $a \cap b = GLB(a, b)$ in brief. We also call them join and meet respectively.

Representation Lattice

1. Hasse diagram

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2. Joint/meet table

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3. *Idempotent:* $a \cap a = a, a \cup a = a$.

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3. *Idempotent:* $a \cap a = a, a \cup a = a.$
4. *Absorption:* $(a \cup b) \cap a = a, (a \cap b) \cup a = a.$

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Property

A lattice could be divided into a join-semilattice and a meet-semilattice.

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Theorem

If L is any set in which there are two operation defined as $\cup$ and $\cap$ satisfying the last four properties, then L is a lattice.

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The subset S of the lattice L is called *convex* if $a, b \in S, c \in L$, and $a \leq c \leq b$ imply that $c \in S$.

Sublattice

Theorem

Given two lattice L *and* L', *a bijection* $f : L \to L'$ *from* L *to* L' *is an isomorphism if and only if* $a \leq b$ *in* L *implies* $f(a) \leq f(b)$ *in* L'.

Next Class

- Special lattices
- Boolean algebra