# Discrete Mathematics

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# Review

- Truth assignment
- Truth valuation
- Tautology
- Consequence

# Outline

- Tableau proof system

# Terminologies

- signed proposition
- entries of the tableau
- atomic tableau

# Tableau

## Definition (Tableaux)

A *finite tableau* is a binary tree, labeled with signed propositions called entries, such that:

1. All atomic tableaux are finite tableaux.
2. If $\tau$ is a finite tableau, $P$ a path on $\tau$, $E$ an entry of $\tau$ occurring on $P$ and $\tau'$ is obtained from $\tau$ by adjoining the unique atomic tableau with root entry $E$ to $\tau$ at the end of the path $P$, then $\tau'$ is also a finite tableau.

If $\tau_0, \tau_1, \ldots, \tau_n, \ldots$ is a (finite or infinite) sequence of the finite tableaux such that, for each $n \geq 0, \tau_{n+1}$ is constructed from $\tau_n$ by an application of (2), then $\tau = \cup \tau_n$ is a *tableau*.

# Tableau

## Example

A tableau with the signed proposition
$F(((\alpha \to \beta) \vee (\gamma \vee \delta)) \wedge (\alpha \vee \beta))$.

# Tableau

## Definition

Let $\tau$ be a tableau, $P$ a path on $\tau$ and $E$ an entry occurring on $P$.

1. $E$ has been *reduced* on $P$ if all the entries on one path through the atomic tableau with root $E$ occur on $P$.
2. $P$ is *contradictory* if, for some proposition $\alpha$, $T\alpha$ and $F\alpha$ are both entries on $P$. $P$ is *finished* if it is contradictory or every entry on $P$ is reduced on $P$.
3. $\tau$ is *finished* if every path through $\tau$ is finished.
4. $\tau$ is *contradictory* if every path through $\tau$ is contradictory.

# Proof

## Definition

1. A *tableau proof* of a proposition $\alpha$ is a contradictory tableau with root entry $F\alpha$. A proposition is *tableau provable*, written $\vdash \alpha$, if it has a tableau proof.
2. A *tableau refutation* for a proposition $\alpha$ is a contradictory tableau starting with $T\alpha$. A proposition is *tableau refutable* if it has a tableau refutation.

# Complete Systematic Tableaux

## Definition (Complete systematic tableaux)

Let $R$ be a signed proposition. We define the *complete systematic tableau*(CST) with root entry $R$ by induction.

1. Let $\tau_0$ be the unique atomic tableau with $R$ at its root.
2. Assume that $\tau_m$ has been defined. Let $n$ be the smallest level of $\tau_m$ and let $E$ be the leftmost such entry of level $n$.
3. Let $\tau_{m+1}$ be the tableau gotten by adjoining the unique atomic tableau with root $E$ to the end of every noncontradictory path of $\tau_m$ on which $E$ is unreduced.

The union of the sequence $\tau_m$ is our desired complete systematic tableau.

# Properties of CST

**Theorem**

*Every CST is finished.*

**Proof.**

Reduce the $E$ level by level and there is no $E$ unreduced for any fixed level. □

# Properties of CST

## Theorem

*If* $\tau = \cup\tau_n$ *is a contradictory tableau, then for some* $m, \tau_m$ *is a finite contradictory tableau. Thus, in particular, if a CST is a proof, it is a finite tableau.*

## Proof.

By König lemma. □

# Properties of CST

## Definition

We define $d(\alpha)$, the degree of a proposition $\alpha$ by induction.

1. if $\alpha$ is a propositional letter, then $d(\alpha) = 0$.
2. if $\alpha$ is $\neg\beta$, then $d(\alpha) = d(\beta) + 1$.
3. if $\alpha$ is $\beta \vee \gamma, \beta \wedge \gamma,$ or $\beta \rightarrow \gamma$, then $d(\alpha) = d(\beta) + d(\gamma) + 1$.

The degree of a signed proposition $T\alpha$ or $F\alpha$ is the degree of $\alpha$. If $P$ is a path in a tableau $\tau$, then $d(P)$ the degree of $P$ is the sum of the degree of the signed propositions on $P$ that are not reduced on $P$.

# Properties of CST

**Theorem**

*Every CST is finite.*

**Proof.**

Every path is finite with $d(P_{m+1}) < d(P_m)$. $\square$

# Next Class

- Soundness theorem
- Completeness theorem