# Discrete Mathematics

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# Review

- Formation tree
- Parsing algorithm

# Outline

- Truth assignment
- Truth valuation
- Tautology
- Consequence

# Truth Assignment

How we discuss the truth of propositional letters?

## Definition (Assignment)

A *truth assignment* $\mathcal{A}$ is a function that assigns to each **propositional letter** $A$ a unique truth value $\mathcal{A}(A) \in \{T, F\}$.

# Truth Valuation

How we discuss the truth of propositions?

## Example

Truth assignment of $\alpha$ and $\beta$ and valuation of $(\alpha \vee \beta)$.

| $\alpha$ | $\beta$ | $(\alpha \vee \beta)$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

# Assignment and Valuation

## Definition (Valuation)

A *truth valuation* $\mathcal{V}$ is a function that assigns to each **proposition** $\alpha$ a unique truth value $\mathcal{V}(\alpha)$ so that its value on a compund proposition is determined in accordance with the appropriate truth tables.

Specially, $\mathcal{V}(\alpha)$ determines one possible *truth assignment* if $\alpha$ is a propositional letter.

# Assignment and Valuation

## Theorem

*Given a truth assignment $\mathcal{A}$ there is a unique truth valuation $\mathcal{V}$ such that $\mathcal{V}(\alpha) = \mathcal{A}(\alpha)$ for every propositonal letter $\alpha$.*

## Proof.

The proof can be divided into two step.

1. Construct a $\mathcal{V}$ from $\mathcal{A}$ by induction on the depth of the associated formation tree.
2. Prove the uniqueness of $\mathcal{V}$ with the same $\mathcal{A}$ by induction bottom-up.

□

# Assignment and Valuation

## Corollary

*If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two valuations that agree on the support of $\alpha$, the finite set of propostional letters used in the construction of the proposition of the proposition $\alpha$, then $\mathcal{V}_1(\alpha) = \mathcal{V}_2(\alpha)$.*

# Tautology

## Definition

A proposition $\sigma$ of propostional logic is said to be *valid* if for any valuation $\mathcal{V}, \mathcal{V}(\sigma) = T$. Such a proposition is also called a *tautology*.

# Tautology

## Example

$\alpha \vee \neg\alpha$ is a tautology.

## Solution:

| $\alpha$ | $\neg\alpha$ | $\alpha \vee \neg\alpha$ |
|---|---|---|
| T | F | T |
| F | T | T |

$\square$

# Logical Equivenlence

## Definition

Two proposition $\alpha$ and $\beta$ such that, for every valuation $\mathcal{V}, \mathcal{V}(\alpha) = \mathcal{V}(\beta)$ are called *logically equivalent*. We denote this by $\alpha \equiv \beta$.

# Logical Equivenlence(Cont.)

## Example

$\alpha \to \beta \equiv \neg\alpha \vee \beta$.

## Proof.

Prove by truth table.

| $\alpha$ | $\beta$ | $\alpha \to \beta$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

| $\alpha$ | $\beta$ | $\neg\alpha$ | $\neg\alpha \vee \beta$ |
|---|---|---|---|
| T | T | F | T |
| T | F | F | F |
| F | T | T | T |
| F | F | T | T |

□

# Consequence

## Definition

Let $\Sigma$ be a (possibly infinite) set of propositions. We say that $\sigma$ is a *consequence* of $\Sigma$ (and write as $\Sigma \models \sigma$) if, for any valuation $\mathcal{V}$,

$$(\mathcal{V}(\tau) = T \text{ for all } \tau \in \Sigma) \Rightarrow \mathcal{V}(\sigma) = T.$$

# Consequence

## Example

1. Let $\Sigma = \{A, \neg A \vee B\}$, we have $\Sigma \models B$.
2. Let $\Sigma = \{A, \neg A \vee B, C\}$, we have $\Sigma \models B$.
3. Let $\Sigma = \{\neg A \vee B\}$, we have $\Sigma \not\models B$.

# Model

## Definition

We say that a valuation $\mathcal{V}$ is a *model* of $\Sigma$ if $\mathcal{V}(\sigma) = T$ for every $\sigma \in \Sigma$. We denote by $\mathcal{M}(\Sigma)$ the set of all models of $\Sigma$.

# Model

## Example

Let $\Sigma = \{A, \neg A \vee B\}$, we have models:

1. Let $\mathcal{A}(A) = T, \mathcal{A}(B) = T$
2. Let $\mathcal{A}(A) = T, \mathcal{A}(B) = T, \mathcal{A}(C) = T$.
3. Let $\mathcal{A}(A) = T, \mathcal{A}(B) = T, \mathcal{A}(C) = F, \mathcal{A}(D) = F, \ldots$.

# Model

## Definition

We say that propositions $\Sigma$ is *satisfiable* if it has some model. Otherwise it is called *unsatisfiable*. To a proposition, it is called *invalid*.

# Properties

## Proposition

*Let $\Sigma, \Sigma_1, \Sigma_2$ be sets of propositions. Let $Cn(\Sigma)$ denote the set of consequence of $\Sigma$ and $Taut$ the set of tautologies.*

1. $\Sigma_1 \subseteq \Sigma_2 \Rightarrow Cn(\Sigma_1) \subseteq Cn(\Sigma_2)$.
2. $\Sigma \subseteq Cn(\Sigma)$.
3. $Taut \subseteq Cn(\Sigma) = Cn(Cn(\Sigma))$.
4. $\Sigma_1 \subseteq \Sigma_2 \Rightarrow \mathcal{M}(\Sigma_2) \subseteq \mathcal{M}(\Sigma_1)$.
5. $Cn(\Sigma) = \{\sigma | \mathcal{V}(\sigma) = T \text{ for all } \mathcal{V} \in \mathcal{M}(\Sigma)\}$.
6. $\sigma \in Cn(\{\sigma_1, \ldots, \sigma_n\}) \Leftrightarrow \sigma_1 \rightarrow (\sigma_2 \ldots \rightarrow (\sigma_n \rightarrow \sigma) \ldots) \in Taut$.

# Deduction Theorem

## Theorem

*For any propositions* $\varphi, \psi$, $\Sigma \cup \{\psi\} \models \varphi \Leftrightarrow \Sigma \models \psi \rightarrow \varphi$ *holds.*

## Proof.

Prove by the definition of consequence.
When we consider $\Rightarrow$, $\mathcal{V}$ which satisfy $\Sigma$ are divided into two parts, $\mathcal{V}_1(\psi) = T$ and $\mathcal{V}_2(\psi) = F$. Then we investigate whether $\mathcal{V}$ satisfies $\psi \rightarrow \varphi$.
Conversely, $\mathcal{V}$ which makes $\psi$ false are discarded. Because they are not taken into consideration to satisfy $\Sigma \cup \{\psi\}$. □

# Next Class

- Tableau proof system