Discrete Mathematics

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Review

- Limits of PL
- Predicates and quantifiers

Outline

- Terms
- Formuals
- Formation tree

Predicates and Quantifiers

- *predicates*
- *variables*
- *constants*
- *functions*
- *universal quantifier*: $\forall$, "for all".
- *existential quantifier*: $\exists$, there exists

Formula

Definition (Subformula)

A *subformula* of a formula φ is a consecutive sequence of symbols from φ which is itself a formula.

Example

Given an formula
$(((\forall x)(\varphi(x) \vee \phi(x, y))) \to ((\exists z)\sigma(z)))$.

1. Is $((\forall x)\varphi(x))$ a subformula?
2. Is $\sigma(x)$ a subformula?
3. $((\forall x)(\varphi(x) \vee \phi(x, y)))$, $((\exists z)\sigma(z))$,$\varphi(x)$, $\phi(x, y)$, and $\sigma(z)$ all are subformulas.

Occurrence

Example

Consider the following examples:

1. $(((\forall x)\varphi(x,y)) \wedge ((\exists x)\psi(x)))$
2. $(((\forall x)\varphi(x,y)) \wedge \psi(x))$

Definition (Occurrence)

An *occurrence* of a variable v in a formula φ is *bound* if there is a subformula ψ of φ containing that occurrence of v such that ψ begins with $((\forall v)$ or $((\exists v)$. An occurrence of v in φ is *free* if it is not bound.

Free Occurrence

Example

Consider the following examples:

1. $((\exists y)((\forall x)\varphi(x, y)) \wedge \psi(x))$

Definition

A variable v is said to *occur free* in φ if it has at least one free occurrence there.

Sentence

Example

Consider the following examples:

1. $((\exists y)((\forall x)\varphi(x,y)) \wedge (\forall z)\psi(z))$
2. $((\forall x)(((\forall y)R(x,y)) \vee ((\exists y)T(x,y)))$.
3. $\varphi(c_0, c_1)$

Definition

A *sentence* of predicate logic is a formula with no free occurrences of any variable.

Open Formula

Definition

An *open formula* is a formula without quantifiers .

Example

1. All atomic formulas: $\phi(x), R(x, y)$ ....
2. $(R(x, y) \vee \phi(x))$.
3. $R(c_0, c_1)$.

Substitution

Definition (Substitution(Instantiation))

If φ is a formula and v a variable, we write $\varphi(v)$ to denote the fact that v occurs free in φ.

1. If t is a term, then $\varphi(t)$, or if we wish to be more explicit, $\varphi(v/t)$, is the result of substituting (or instantiating) t for all free occurrences of v in φ. We call $\varphi(t)$ an *instance* of φ.
2. If $\varphi(t)$ contains no free variables, we call it a *ground instance* of φ.

Substitution

Example

Given a formula $((\forall x)R(x,y)) \vee ((\exists y)S(x,y))$,

1. It is denoted as $\varphi(x,y)$.
2. $\varphi(x/s, y/t) = ((\forall x)R(x,t)) \vee ((\exists y)S(s,y))$.
3. $\varphi(x/c, y/d) = ((\forall x)R(x,d)) \vee ((\exists y)S(c,y))$.

Substitution

Example

The recursive definition of $\varphi(x/t)$ with substitution t_0.

The recursive definition of t_0 to x in t.

1. t is a constant, $t[x/t_0] = t$
2. t is a variable,
$$t[x/t_0] = \begin{cases} t_0 & \text{if } t = x \\ t & \text{o.w. } (t \neq x) \end{cases}$$
3. $t = g(t_1, t_2, \ldots, t_k)$,
$$t[x/t_0] = g(t_1[x/t_0], t_2[x/t_0], \ldots, t_k[x/t_0])$$

Substitution

Example (Cont.)

1. $\varphi(x, y)$ is an atomic formula, $\varphi[x/t_0] = \varphi(t_0, y)$
2. $\varphi = \neg\psi$, $\varphi[x/t_0] = \neg\psi[x/t_0]$
3. $\varphi = \varphi_1 \Box \varphi_2$, $\varphi[x/t_0] = \varphi_1[x/t_0] \Box \varphi_2[x/t_0]$
4. $\varphi(x, y) = Q_z\psi(x, y, z)$,

$$\varphi[x/t_0] = \begin{cases} \varphi, & x = z \\ Q_z(\psi[x/t_0]), & x \neq z \end{cases}$$

Substitution

Definition

If the term t contains an occurrence of some variable x (which is necessarily free in t) we say that t is *substitutable* for the free variable v in $\varphi(v)$ if all occurrences of x in t remain free in $\varphi(v/t)$.

Substitution

Example

Let $\varphi(x) = (((\exists y)R(x,y)) \vee ((\forall z)\neg Q(x,z)))$.

1. If $t = f(w,u)$, then we have $\varphi(t) = \varphi(x/t) = (((\exists y)R(f(w,u),y)) \vee ((\forall z)\neg Q(f(w,u),z)))$.
2. If $t = g(y, s(y))$, it is not substitutable for x in $\varphi(x)$.

Properties of term

Proposition

If a term s is an initial segment of a term $t, s \subseteq t$, then $s = t$.

Theorem (Unique readability for terms)

Every term s is either a variable or constant symbol or of the form $f(s_1, \ldots, s_n)$ in which case f, n and the s_i for $1 \leq i \leq n$ are all unique determined.

Properties of formula

Proposition

If a formula α *is an initial segment of a formula* $\gamma, \alpha \subset \gamma$*, then* $\alpha = \gamma$*.*

Theorem (Unique readability for formulas)

Each formula ϕ *is a precisely one of the following forms: an atomic formula,*
$(\alpha \wedge \beta), (\alpha \rightarrow \beta), (\alpha \leftrightarrow \beta), (\neg\alpha), (\alpha \vee \beta)$*. Moreover, the relevant "components" of* ϕ *as displayed in each of these formula are uniquely determined.*

Formation Trees

Definition

1. *Term formation trees* are ordered, finitely branching tree T labeled with terms satisfying the following conditions:
    1. The leaves of T are labeled with variables of constant symbols.
    2. Each nonleaf node of T is labeled with a term of the form $f(t_1, \ldots, t_n)$.
    3. A node of T that is labeled with a term of the form $f(t_1, \ldots, t_n)$ has exactly n immediate successors in the tree. They are labeled in (lexicographic) order with $t_1, \ldots, t_n$.
2. A term formation tree is *associated with* the term with which its root node is labeled.

Term and Formation Tree

Proposition

Every term t has a unique formation tree associated with it.

Proposition

The ground terms are those terms whose formation trees have no variables on their leaves.

Formation Trees(Cont.)

Definition

The *atomic formula auxiliary formation trees* are the labeled , ordered, finitely branching trees of depth one whose root node is labeled with an atomic formula. If the root node of such a tree is labeled with an n-ary relation $R(t_1, \ldots, t_n)$, then it has n immediate successor which are labeled in order with the terms $t_1, \ldots, t_n$.

Formation Trees(Cont.)

Definition

The *atomic formula formation trees* are the finitely branching, labeled, ordered trees gotten from the auxiliary trees by attaching at each leaf labeled with a term the rest of the formation tree associated with t. Such a tree is associated with the atomic formula with which its root is labeled.

Formation Trees(Cont.)

Proposition

Every atomic formula is associated with a unique formation tree.

Formation Trees(Cont.)

Definition

The *formula auxiliary formation trees* are the labeled, ordered, binary branching trees T such that

1. The leaves of T are labeled with atomic formulas.
2. If σ is a nonleaf node of T with one immediate successor $\sigma \wedge 0$ labeled with a formula φ, then σ is labeled with $\neg\varphi, \exists v\varphi,$ or $\forall v\varphi$ for some variable v.
3. If σ is a nonleaf node with two immediate successors, $\sigma \wedge 0$ and $\sigma \wedge 1$ labeled with formulas φ and ψ, then σ is labeled with $\varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \varphi \leftrightarrow \psi$.

Formation Trees(Cont.)

Definition

1. The *formula formation trees* are the ordered, labeled trees gotten from the auxiliary ones by attaching to each leaf labeled with an atomic formula the rest of its associated formation tree. Each such tree is again associated with the formula with which its root is labeled.
2. The *depth of a formula* is the depth of the associated auxiliary formation tree.

Formula and Formation Tree

Proposition

Every formula is associated with a unique(auxiliary) formation tree.

Proposition

The subformulas of a formula φ are the labels of the nodes of the auxiliary formation tree associated with φ.

Formula and Formation Tree(Cont.)

Proposition

1. *The occurrences of a variable v in a formula φ are in one-one correspondence with the leaves of the associated formation tree that are labeled with v. We may also refer to the appropriate leaf labeled with v as the occurrence of v in φ.*
2. *An occurrence of the variable v in φ is bound if there is a formula ϕ beginning with $((\forall v)$ or $((\exists v)$ which is the label of a node above the corresponding leaf of the formation tree for φ labeled with v.*

Formula and Formation Tree(Cont.)

Proposition

If φ is a formula and v a variable, then $\varphi(v/t)$ is the formula associated with the formation tree gotten by replacing each leaf in the tree for $\varphi(v)$ which is labeled with a free occurrence of v with the formation tree associated with t and propagating this change through the tree.

Formula and Formation Tree(Cont.)

Proposition

The term t is substitutable for v in $\varphi(v)$ if all occurrences of x in t remain free in $\varphi(t)$, i.e., any leaf in the formation tree for t which is a free occurrence of a variable x remains in every location in which it appears in the formation tree described in Definition 3.8.

Parsing Algorithm

To recognize a formula, we should pay attention to

1. Two different categories of objects:
   1. atomic formula.
   2. term.
2. Nested terms.
3. Number of parameters.

Next Class

- Structure
- Interpretation
- Truth
- Satisfiable
- Consequence