Discrete Mathematics

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April 23, 2013

Review

- Soundness
- Completeness

Outline

- Deduction from premises
- Compactness
- Applications

Consequence

Definition

Let Σ be a (possibly infinite) set of propositions. We say that σ is a *consequence* of Σ (and write as $\Sigma \models \sigma$) if, for any valuation $\mathcal{V}$,

$$(\mathcal{V}(\tau) = T \text{ for all } \tau \in \Sigma) \Rightarrow \mathcal{V}(\sigma) = T.$$

Consequence

Example

1. Let $\Sigma = \{A, \neg A \vee B\}$, we have $\Sigma \models B$.
2. Let $\Sigma = \{A, A \to B\}$, we have $\Sigma \models B$.
3. Let $\Sigma = \{\neg A\}$, we have $\Sigma \models (A \to B)$.

Deductions from Premises

How to construct CST from premises?

Definition (Tableaux from premises)

Let Σ be (possibly infinite) set of propositions. We define the *finite tableaux with premises from* Σ by induction:

2. If τ is a finite tableau from Σ and $\alpha \in \Sigma$, then the tableau formed by putting $T\alpha$ at the end of every noncontradictory path not containing it is also a finite tableau from Σ.

Tableau proof

Definition

A *tableau proof of a proposition* α *from* Σ is a tableau from Σ with root entry $F\alpha$ that is contradictory, that is, one in which every path is contradictory. If there is such a proof we say that α is *provable from* Σ and write it as $\Sigma \vdash \alpha$.

Property of CST

Theorem

Every CST from a set of premises is finished.

Soundness of deductions from premises

Theorem

If there is a tableau proof of α from a set of premises Σ, then α is a consequence of Σ, i.e. $\Sigma \vdash \alpha \Rightarrow \Sigma \models \alpha$.

Completeness of deduction from premises

Theorem

If α is consequence of a set Σ of premises, then there is a tableau deduction of α from Σ, i.e., $\Sigma \models \alpha \Rightarrow \Sigma \vdash \alpha$.

Finite proof

Theorem

If $\tau = \cup\tau_n$ is a contradictory tableau from Σ, then for some m, τ_m is a finite contradictory tableau from Σ. In particular, if a CST from Σ is a proof, it is finite.

Compactness

Theorem

α is a consequence of Σ if and only if α is a consequence of some finite subset of Σ.

Definition

A set Σ of propositions is called *satisfiable* if it has a model, i.e., there is a valuation $\mathcal{V}$ such that $\mathcal{V}(\alpha) = T$ for every $\alpha \in \Sigma$. We also say that such a valuation *satisfies* Σ.

Compactness

Theorem (Compactness)

Let $\Sigma = \{\alpha_i | i \in \omega\}$ *be an infinite set of a propositions.* Σ *is satisfiable if and only if every finite subset* Γ *of* Σ *is satisfiable.*

Model check: problem

Example

There are three suspects A, B, and C who are associated with a murder case. The police officer queried them and have their statements:

- A: I didn't kill the victim, he is a friend of B and C hates him.
- B: I didn't do it. Even I don't know him. And I am not present.
- C: I didn't do it. I saw A and B stayed with the victim in that day. The murder must be one of them.

Our question is who the suspect is?

Model check: solution

We can assume that the murder would lie to cover his action. So we want check the truth of statements. So we have:

1. A: A killed victim.
2. BKV: B knows the victim.
3. AP: A is present.
4. CHV: C hates the victim.
5. $(A \wedge \neg B) \vee (\neg A \wedge B)$: murder is either A or B.

Model check: solution

Now we can represent the satement of each sucpects as following:

1. A: $\neg A \wedge BKV \wedge CHV$.
2. B: $\neg B \wedge \neg BKV \wedge \neg BP$.
3. C: $\neg C \wedge AP \wedge BP \wedge ((A \wedge \neg B) \vee (\neg A \wedge B))$.

It is easy to find the maximal satisfiable set of propositions.

Digital design

Example

Consider the circuit for the following propositions:

1. $(A_1 \wedge A_2) \vee (\neg A_3))$
2. $(A \wedge B \wedge D) \vee (A \wedge B \wedge \neg C)$

Digital design

Example

Consider the boolean function majority of $\{A, B, C\}$. It means that the value of function depends on the majority of input.

We can use a proposition to represent majority function:

$$
\begin{aligned}
m(A, B, C) &= (A \wedge B \wedge C) \vee (A \wedge B \wedge \neg C) \\
&\vee (A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge C) \\
&= (B \wedge C) \vee (A \wedge C) \vee (A \wedge B) \\
&= (A \wedge (B \vee C)) \vee (B \wedge C)
\end{aligned}
$$

Application of proposition logic

Example

Consider the pigeonhole principle:
$f : n^+ \rightarrow n, \exists i, j, f(i) = f(j),$ where $0 \leq i < j \leq n$.

Application of proposition logic

Solution: let p_{ij} means $f(i) = j$. Then we can describe everywhere defined property as

$$\alpha_1 = \bigwedge_{0 \le i \le n} \bigvee_{0 \le j < n} p_{ij}$$

and we can describe single value as

$$\alpha_2 = \bigwedge_{0 \le i \le n} \bigwedge_{0 \le j \ne k < n} \neg(p_{ij} \wedge p_{ik})$$

Now we can describe pigeonhole principle as

$$\varphi = (\alpha_1 \wedge \alpha_2) \wedge \bigvee_{0 \le i < j \le n} \bigvee_{0 \le k < n} (p_{ik} \wedge p_{jk})$$

Application of compactness theorem

Example

Given an infinite planar graph. If its every finite subgraph is k-colorable, then the graph itself is also k-colorable.

Application of compactness theorem

Solution: Let $p_{a,i}$ represent vertex a is colored with i. We can formulate a graph which is k-colorable with the following propositions.

1. $p_{a,1} \vee p_{a,2} \vee \cdots \vee p_{a,k}$, for every $a \in V$. It means every vertex could be colored with at least one of k colors.
2. $\neg(p_{a,i} \wedge p_{a,j}), 1 \leq i < j \leq k$ for all $a \in V$. It means $C_i \cap C_j = \emptyset$.
3. $\neg(p_{a,i} \wedge p_{b,i}), i = 1, \ldots, k$ for all aEb. It means no neighbors have the same color .

Application of compactness theorem

Example

Every set S can be (totally) ordered.

Application of compactness theorem

Theorem

An infinite tree with finite branch has an infinite path.

Proof.

Read lecture note. □

Next Class

- Middle Term Examination, 1:30 - 3:30, May 5.
- Predicate Logic, May 7